

AN ABELIAN QUOTIENT OF THE SYMPLECTIC DERIVATION LIE ALGEBRA OF THE FREE LIE ALGEBRA

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ABSTRACT. We construct an abelian quotient of the symplectic derivation Lie algebra $\mathfrak{h}_{g,1}$ of the free Lie algebra generated by the fundamental representation of $\mathrm{Sp}(2g, \mathbb{Q})$. More specifically, we show that the weight 12 part of the abelianization of $\mathfrak{h}_{g,1}$ is 1-dimensional for $g \geq 8$. The computation is done with the aid of computers.

1. INTRODUCTION

Let H be the fundamental representation over \mathbb{Q} of the symplectic group $\mathrm{Sp}(2g, \mathbb{Q})$. Topologically, the vector space H is the first rational homology group of a compact connected oriented surface of genus g with one boundary component. In this paper, we concern with the symplectic derivation Lie algebra $\mathfrak{h}_{g,1}$ of the free Lie algebra $\mathcal{L}(H)$ generated by H . The Lie algebra $\mathfrak{h}_{\infty,1} := \lim_{g \rightarrow \infty} \mathfrak{h}_{g,1}$ obtained by the stabilization is just the *Lie case* of the three infinite-dimensional graded Lie algebras considered by Kontsevich in [12, 13]. In these papers, he proved that the Lie algebra homology of $\mathfrak{h}_{\infty,1}$ is isomorphic to the free graded commutative algebra generated by the stable homology of the Lie algebra $\mathfrak{sp}(2h, \mathbb{Q})$ of $\mathrm{Sp}(2h, \mathbb{Q})$ and the totality of the cohomology of the outer automorphism groups $\mathrm{Out} F_n$ of free groups of rank $n \geq 2$.

In general, computing (co)homology groups of $\mathrm{Out} F_n$ has been a difficult problem. More specifically, although the theory of outer spaces due to Culler and Vogtmann [7] gives a $(2n - 3)$ -dimensional finite cell complex which computes the rational (co)homology groups of $\mathrm{Out} F_n$ for any fixed n , the number of cells grows very fast comparing with increasing n . $H_*(\mathrm{Out} F_n; \mathbb{Q})$ were determined by Hatcher and Vogtmann [11] for $n \leq 4$ (together with general computational results), Gerlits [9] for $n = 5$, and Ohashi [21] for $n = 6$. On the other hand, in our previous papers [18, 19], we computed the integral Euler characteristics

$$e(\mathrm{Out} F_n) = \sum_{i=0}^{2n-3} (-1)^i \dim (H_i(\mathrm{Out} F_n; \mathbb{Q}))$$

of $\mathrm{Out} F_n$ up to $n \leq 11$:

n	2	3	4	5	6	7	8	9	10	11
$e(\mathrm{Out} F_n)$	1	1	2	1	2	1	1	-21	-124	-1202

This result shows the existence of many non-trivial *odd*-dimensional rational (co)homology classes of $\mathrm{Out} F_n$. Note that almost all of the above results were obtained with the aid of computers.

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Recently, Bartholdi [1] computed the rational homology of $\text{Out } F_7$. The computation was also aided by computers. After a big calculation, he obtained

$$H_i(\text{Out } F_7; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & (i = 0, 8, 11) \\ 0 & (\text{otherwise}) \end{cases}.$$

In this result, $H_{11}(\text{Out } F_7; \mathbb{Q}) \cong \mathbb{Q}$ is remarkable because it is the first non-trivial *odd* and (*virtually*) *top* rational homology group which is explicitly described. Here recall that the complex constructed by Culler and Vogtmann is $(2n - 3)$ -dimensional, which is known to give the virtual cohomological dimension of $\text{Out } F_n$. It is a sharp contrast with the fact that the virtually top rational homology groups vanish for related groups such as $\text{GL}(n, \mathbb{Z})$ and $\text{SL}(n, \mathbb{Z})$ (see Lee-Szczarba [14]), and the mapping class groups of surfaces of genus $g \geq 2$ with at most one puncture (see our paper [17] or Church-Farb-Putman [2], see also Conant-Kassabov-Vogtmann [5]).

By applying Kontsevich's theorem to the above fact that $H_{11}(\text{Out } F_7; \mathbb{Q}) \cong \mathbb{Q}$, we have $H_1(\mathfrak{h}_{\infty,1})_{12} \cong \mathbb{Q}$, where $H_1(\mathfrak{h}_{\infty,1})_{12}$ is the weight 12 part of the abelianization of the graded Lie algebra $\mathfrak{h}_{\infty,1}$ (see Section 2 for details). The purpose of the present paper is to give an explicit description of this isomorphism purely in terms of the Lie algebra $\mathfrak{h}_{\infty,1}$. Consequently, we give an alternative proof of $H_{11}(\text{Out } F_7; \mathbb{Q}) \cong \mathbb{Q}$ by a different method. Our computation is also aided by computers. In general, it is desirable for a computer-aided result to be checked by multiple methods.

Our computation of $H_1(\mathfrak{h}_{\infty,1})_{12}$ is given by calculating $H_1(\mathfrak{h}_{g,1})_{12}$ for sufficiently large g . We construct an $\text{Sp}(2g, \mathbb{Q})$ -invariant cocycle

$$C : \mathfrak{h}_{g,1}(12) \longrightarrow \mathbb{Q}$$

and show that it is non-trivial for all $g \geq 2$. This means that $H_1(\mathfrak{h}_{g,1})_{12}$ is non-trivial even for g in the unstable range and we finally show that C gives an isomorphism $H_1(\mathfrak{h}_{g,1})_{12} \cong \mathbb{Q}$ for $g \geq 8$. Note that the non-triviality of $H_1(\mathfrak{h}_{g,1})_{12}$ in the unstable range was unknown, and in the paper [16] by Gwénaél Massuyeau and the second named author, we give a topological application of this fact. The full description for C is put in Appendix. The authors are trying to understand the meaning of our cocycle C with a hope to generalize it in higher weights (see Section 5), although it seems difficult because the cocycle is very big and complicated.

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2. THE SYMPLECTIC DERIVATION LIE ALGEBRA OF THE FREE LIE ALGEBRA

The fundamental representation H of $\text{Sp}(2g, \mathbb{Q})$ is $2g$ -dimensional and has a natural non-degenerate anti-symmetric bilinear form

$$\mu : H \otimes H \longrightarrow \mathbb{Q}.$$

We identify H with the dual space H^* by μ . Fix a symplectic basis $\{a_1, b_1, \dots, a_g, b_g\}$ of H with respect to μ . That is, they satisfy

$$\mu(a_i, a_j) = \mu(b_i, b_j) = 0, \quad \mu(a_i, b_j) = \delta_{i,j}$$

for any $1 \leq i, j \leq g$.

The free Lie algebra $\mathcal{L}(H)$ generated by H has a natural graded Lie algebra structure. Let

$$\mathcal{L}(H) = \bigoplus_{i=1}^{\infty} \mathcal{L}_i(H)$$

denote the corresponding direct sum decomposition with $\mathcal{L}_i(H)$ the degree i part. We now consider the space $\mathfrak{h}_{g,1}$ of all derivations of $\mathcal{L}(H)$ that annihilate the symplectic element $\omega_0 := \sum_{i=1}^g [a_i, b_i] \in \mathcal{L}_2(H)$. That is, an endomorphism D of $\mathcal{L}(H)$ is in $\mathfrak{h}_{g,1}$ if it satisfies the *Leibniz rule*

$$D([X, Y]) = [D(X), Y] + [X, D(Y)]$$

for all $X, Y \in \mathcal{L}(H)$ and $D(\omega_0) = 0$. $\mathfrak{h}_{g,1}$ is a Lie subalgebra of the Lie algebra of all endomorphisms of $\mathcal{L}(H)$. Let $\mathfrak{h}_{g,1}(k)$ be the degree k homogeneous part of $\mathfrak{h}_{g,1}$. Since the Leibniz rule says that each element of $\mathfrak{h}_{g,1}$ is characterized by its action on $\mathcal{L}_1(H) = H$, we have

$$\mathfrak{h}_{g,1}(k) = \{D \in \mathfrak{h}_{g,1} \mid D(H) \subset \mathcal{L}_{k+1}(H)\}.$$

In other words, $\mathfrak{h}_{g,1}(k)$ is regarded as a subspace of

$$\text{Hom}(H, \mathcal{L}_{k+1}(H)) = H^* \otimes \mathcal{L}_{k+1}(H) = H \otimes \mathcal{L}_{k+1}(H).$$

From this point of view, it is known that

$$\mathfrak{h}_{g,1}(k) = \text{Ker} \left(H \otimes \mathcal{L}_{k+1}(H) \xrightarrow{[\cdot, \cdot]} \mathcal{L}_{k+2}(H) \right)$$

and in particular, $\mathfrak{h}_{g,1}(0)$ coincides with the Lie algebra $\mathfrak{sp}(2g, \mathbb{Q})$ of $\text{Sp}(2g, \mathbb{Q})$. It is easy to check that the direct sum decomposition

$$\mathfrak{h}_{g,1} = \bigoplus_{k=0}^{\infty} \mathfrak{h}_{g,1}(k)$$

gives a graded Lie algebra structure. We call $\mathfrak{h}_{g,1}$ the *symplectic derivation Lie algebra of $\mathcal{L}(H)$* . The symplectic group $\text{Sp}(2g, \mathbb{Q})$ acts naturally on each $\mathfrak{h}_{g,1}(k)$. This action is the restriction of the diagonal action on $H \otimes \mathcal{L}_{k+1}(H)$, and therefore it is compatible with the stabilization $\mathfrak{h}_{g,1} \hookrightarrow \mathfrak{h}_{g+1,1}$.

In this paper, we concern with the abelianization $H_1(\mathfrak{h}_{g,1}) = \mathfrak{h}_{g,1}/[\mathfrak{h}_{g,1}, \mathfrak{h}_{g,1}]$ of the Lie algebra $\mathfrak{h}_{g,1}$ as well as $H_1(\mathfrak{h}_{\infty,1}) = \mathfrak{h}_{\infty,1}/[\mathfrak{h}_{\infty,1}, \mathfrak{h}_{\infty,1}]$ after taking the direct limit with respect to g . The grading of $\mathfrak{h}_{g,1}$ gives a direct sum decomposition

$$H_1(\mathfrak{h}_{g,1}) = \bigoplus_{w=0}^{\infty} H_1(\mathfrak{h}_{g,1})_w$$

with

$$H_1(\mathfrak{h}_{g,1})_w := \mathfrak{h}_{g,1}(w) / \sum_{i=0}^w [\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(w-i)]$$

called the *weight w part*. Now we recall the theorem of Kontsevich mentioned in Introduction. For simplicity, we only mention the part related to $H_1(\mathfrak{h}_{\infty,1})$ and refer to the original papers [12, 13] and a paper by Conant and Vogtmann [6] for details. In these papers, it is shown that there exists an isomorphism

$$H_1(\mathfrak{h}_{\infty,1})_{2n} \cong H^{2n-1}(\text{Out } F_{n+1}; \mathbb{Q})$$

for any integer $n \geq 1$. Here $\text{Out } F_{n+1}$ is the outer automorphism group of the free group of rank $(n+1)$ and $H^{2n-1}(\text{Out } F_{n+1}; \mathbb{Q})$ denotes the $(2n-1)$ -st rational cohomology group of $\text{Out } F_{n+1}$. By a technical reason, we rewrite $H_1(\mathfrak{h}_{\infty,1})_{2n}$ as follows. Let

$$\mathfrak{h}_{g,1}^+ = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$$

be the ideal of the *positive* degree part. The spaces $H_1(\mathfrak{h}_{g,1})_w$ and $H_1(\mathfrak{h}_{g,1}^+)_w$ are both $\text{Sp}(2g, \mathbb{Q})$ -modules. As shown in the above cited papers (see also [18, Section 2]), we have

$$H_1(\mathfrak{h}_{g,1})_w \cong H_1(\mathfrak{h}_{g,1}^+)_w^{\text{Sp}} = \mathfrak{h}_{g,1}(w)^{\text{Sp}} / \left(\sum_{i=1}^{w-1} [\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(w-i)] \right)^{\text{Sp}}$$

for any $w \geq 1$. Here, for an $\text{Sp}(2g, \mathbb{Q})$ -module V , we denote by V^{Sp} the invariant subspace for the $\text{Sp}(2g, \mathbb{Q})$ -action and by V_{Sp} the coinvariant quotient of V . The general theory of $\text{Sp}(2g, \mathbb{Q})$ -representations says that for a finite-dimensional representation V , V^{Sp} and V_{Sp} are isomorphic. Hence $\mathfrak{h}_{g,1}^{\text{Sp}} \cong (\mathfrak{h}_{g,1})_{\text{Sp}}$. It is also known that $H_*(\mathfrak{h}_{g,1}^+)_w^{\text{Sp}} \cong (H_*(\mathfrak{h}_{g,1}^+)_w)_{\text{Sp}}$ stabilize when g becomes large. In particular, $H_1(\mathfrak{h}_{\infty,1})_{2n}$ is *finite-dimensional*.

3. MAIN RESULTS

The main result of this paper is to derive $H_1(\mathfrak{h}_{\infty,1})_{12}^{\text{Sp}} \cong \mathbb{Q}$ without using theorems of Kontsevich and Bartholdi. That is, we prove it directly in $\mathfrak{h}_{\infty,1}^+$. In the proof, we construct an explicit linear map (cocycle) C which gives the above isomorphism. More precisely, we have:

Theorem 3.1. *There exists an $\text{Sp}(2g, \mathbb{Q})$ -invariant linear map $C : \mathfrak{h}_{g,1}(12) \rightarrow \mathbb{Q}$ satisfying that*

- *C is non-trivial for any $g \geq 2$,*
- *the restriction of C to $\sum_{i=1}^{11} [\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(12-i)]$ is trivial.*

That is, the cocycle C gives a surjection $\tilde{C} : H_1(\mathfrak{h}_{g,1})_{12}^{\text{Sp}} \cong (H_1(\mathfrak{h}_{g,1})_{12})_{\text{Sp}} \twoheadrightarrow \mathbb{Q}$ for every $g \geq 2$. Moreover \tilde{C} is an isomorphism for $g \geq 8$.

This theorem gives an alternative proof of $H^{11}(\text{Out } F_7; \mathbb{Q}) \cong \mathbb{Q}$.

Remark 3.2. Since $\mathfrak{h}_{1,1}(12)^{\text{Sp}} = 0$ as mentioned in [20], we have $H_1(\mathfrak{h}_{1,1})_{12}^{\text{Sp}} = 0$. Therefore our bound of genus for the non-triviality of $H_1(\mathfrak{h}_{g,1})_{12}^{\text{Sp}}$ is best possible.

4. METHOD FOR COMPUTATION

In this section, we explain how we prove the main theorem with the aid of computers. Our computation for $H_1(\mathfrak{h}_{g,1}^+)_{12}^{\text{Sp}}$ proceeds in the following way:

- (1) Find a coordinate system of $\mathfrak{h}_{g,1}(12)^{\text{Sp}} \cong \mathbb{Q}^{650}$.
- (2) Compute the bracket map

$$[\cdot, \cdot] : \left(\bigoplus_{i=1}^{11} (\mathfrak{h}_{g,1}(i) \otimes \mathfrak{h}_{g,1}(12-i)) \right)^{\text{Sp}} \longrightarrow \mathfrak{h}_{g,1}(12)^{\text{Sp}}$$

and see that the image includes a 649-dimensional subspace W .

- (3) Find a linear map $C : \mathfrak{h}_{g,1}(12)^{\text{Sp}} \twoheadrightarrow \mathbb{Q}$ which annihilates W .
- (4) Check that C is trivial on the image of the bracket map.

4.1. Coordinate system of $\mathfrak{h}_{g,1}(12)^{\text{Sp}}$. We begin by finding out a coordinate system of $\mathfrak{h}_{g,1}(12)^{\text{Sp}} \cong \mathfrak{h}_{g,1}(12)_{\text{Sp}}$, which is known to be isomorphic to \mathbb{Q}^{650} for $g \geq 5$ (see [20, Table 3], where $\dim(\mathfrak{h}_{3,1}(12)^{\text{Sp}})$ is wrongly written, the correct number is 354). Every $\text{Sp}(2g, \mathbb{Q})$ -invariant linear map $\mathfrak{h}_{g,1}(12) \rightarrow \mathbb{Q}$ factors through $\mathfrak{h}_{g,1}(12)_{\text{Sp}}$ and the natural projection $\mathfrak{h}_{g,1}(12) \rightarrow \mathfrak{h}_{g,1}(12)_{\text{Sp}}$ is regarded as the projection onto $\mathfrak{h}_{g,1}(12)^{\text{Sp}}$. Therefore we get a coordinate system of $\mathfrak{h}_{g,1}(12)^{\text{Sp}}$ by finding 650 linearly independent $\text{Sp}(2g, \mathbb{Q})$ -invariant linear maps $\mathfrak{h}_{g,1}(12) \rightarrow \mathbb{Q}$.

Since $\mathfrak{h}_{g,1}(12)$ is an $\text{Sp}(2g, \mathbb{Q})$ -submodule of $H \otimes \mathcal{L}_{13}(H) \subset H^{\otimes 14}$, we have $\mathfrak{h}_{g,1}(12)^{\text{Sp}} \subset (H^{\otimes 14})^{\text{Sp}}$. A coordinate for $(H^{\otimes 14})^{\text{Sp}}$ is classically known and it is given as follows. Divide the set $\{a, b, c, \dots, m, n\}$ of 14 letters into 7 pairs, say $(i_1 j_1)(i_2 j_2) \cdots (i_7 j_7)$. Then we consider the map

$$\mu_{(i_1 j_1)(i_2 j_2) \cdots (i_7 j_7)} : H^{\otimes 14} \rightarrow \mathbb{Q}$$

defined by

$$x_a \otimes x_b \otimes \cdots \otimes x_n \mapsto \mu(x_{i_1}, x_{j_1}) \mu(x_{i_2}, x_{j_2}) \cdots \mu(x_{i_7}, x_{j_7}).$$

Here we call this map a *multiple contraction*. It is $\text{Sp}(2g, \mathbb{Q})$ -invariant since μ is so. We use multiple contractions restricted to $\mathfrak{h}_{g,1}(12)$ as coordinates of $\mathfrak{h}_{g,1}(12)^{\text{Sp}}$. Note that they are invariant under the stabilization map $\mathfrak{h}_{g,1} \hookrightarrow \mathfrak{h}_{g+1,1}$.

On the other hand, we use Lie spiders to express elements of $\mathfrak{h}_{g,1}$. A *Lie spider with $(k+2)$ legs* is defined by

$$\begin{aligned} & S(u_1, u_2, u_3, \dots, u_{k+2}) \\ &:= u_1 \otimes [u_2, [u_3, [\cdots [u_{k+1}, u_{k+2}] \cdots]]] + u_2 \otimes [[u_3, [u_4, [\cdots [u_{k+1}, u_{k+2}] \cdots]]], u_1] \\ &+ u_3 \otimes [[u_4, [u_5, [\cdots [u_{k+1}, u_{k+2}] \cdots]]], [u_1, u_2]] + \cdots + u_{k+2} \otimes [[[\cdots [u_1, u_2], \cdots], u_k], u_{k+1}], \end{aligned}$$

where $u_i \in H$. It is known (see [15], for instance) that Lie spiders with $(k+2)$ legs belong to $\mathfrak{h}_{g,1}(k)$ and generate it.

After calculating the pairings of a large amount of multiple contractions and Lie spiders, we see that the 650 multiple contractions C_1, C_2, \dots, C_{650} in Appendix A are linearly independent. (Here we omit to display the corresponding 650 Lie spiders since we may use the data in Appendix B together with the Lie spider in Subsection 4.4.)

4.2. Computation of the bracket map.

Since the bracket map

$$[\cdot, \cdot] : \bigoplus_{i=1}^{11} (\mathfrak{h}_{g,1}(i) \otimes \mathfrak{h}_{g,1}(12-i)) \longrightarrow \mathfrak{h}_{g,1}(12)$$

is $\mathrm{Sp}(2g, \mathbb{Q})$ -equivariant, it induces a linear map

$$[\cdot, \cdot] : \left(\bigoplus_{i=1}^{11} (\mathfrak{h}_{g,1}(i) \otimes \mathfrak{h}_{g,1}(12-i)) \right)^{\mathrm{Sp}} \longrightarrow \mathfrak{h}_{g,1}(12)^{\mathrm{Sp}}.$$

Recall that each of multiple contractions $\mu_{(i_1 j_1)(i_2 j_2) \dots (i_7 j_7)} : \mathfrak{h}_{g,1}(12) \rightarrow \mathbb{Q}$ factors through $\mathfrak{h}_{g,1}(12)_{\mathrm{Sp}} \cong \mathfrak{h}_{g,1}(12)^{\mathrm{Sp}}$. Therefore as long as we use the coordinate system constructed above, we may work in the whole $\mathfrak{h}_{g,1}(12)$ without considering the projection onto the $\mathrm{Sp}(2g, \mathbb{Q})$ -invariant part.

Although the general formula for the bracket map is a little bit complicated, there is a clear formula for the bracket of two Lie spiders, see [6, Section 2.4.1] for instance. After implementing a computer code for the formula, we compute the coordinates of the images of many brackets in $\mathfrak{h}_{g,1}(12)_{\mathrm{Sp}}$. As a result, we see that the 649 elements of $\sum_{i=1}^{11} [\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(12-i)]$ exhibited in Appendix B generate a 649-dimensional subspace W in $\mathfrak{h}_{g,1}(12)_{\mathrm{Sp}} \cong \mathbb{Q}^{650}$. The result is valid for $g \geq 8$ since we only use $a_1, b_1, \dots, a_8, b_8$ in this computation.

4.3. The map C . Next we look for a non-trivial $\mathrm{Sp}(2g, \mathbb{Q})$ -invariant linear map $C : \mathfrak{h}_{g,1}(12) \rightarrow \mathbb{Q}$ annihilating W . For that we compute a linear relation over $W \cong \mathbb{Q}^{649}$ among our 650 multiple contractions. The result is given by a linear combination of 647 multiple contractions as in Appendix C.

4.4. Checking that C is a non-trivial cocycle. The final step needs the heaviest computation. To show that C is a cocycle, namely it is trivial on the image of the bracket map, we compute general formulas of bracket maps from each of $[\mathfrak{h}_{g,1}(1), \mathfrak{h}_{g,1}(11)]$, $[\mathfrak{h}_{g,1}(2), \mathfrak{h}_{g,1}(10)]$, \dots , $[\mathfrak{h}_{g,1}(6), \mathfrak{h}_{g,1}(6)]$ to $\mathfrak{h}_{g,1}(12)_{\mathrm{Sp}}$ in terms of μ . The results are 0 for all cases. This shows that C is a cocycle for all $g \geq 2$. The non-triviality of C follows from the explicit computation that

$$C(S(a_1, b_1, a_1, a_1, a_1, a_1, a_2, a_1, b_1, b_1, b_1, b_1, b_2)) = 5832,$$

which is valid for all $g \geq 2$.

4.5. Implementation. The authors performed the above computations by using Mathematica. For that they wrote Mathematica codes whose core part computes the pairings of an element in $H^{\otimes 14}$ with multiple contractions. This is not so a difficult task if we use general *transformation (replacement) rules to symbolic expressions* in Mathematica. For example, an element $x_1 \otimes x_2 \otimes \dots \otimes x_{14} \in H^{\otimes 14}$ with $x_i \in \{a_1, b_1, \dots, a_g, b_g\}$ can be represented by the value `ts[x1, x2, ..., x14]` of a function `ts` with *no* definition assigned. We can treat general elements in $H^{\otimes 14}$ by just taking their linear combinations. Then the pairing of $x_1 \otimes x_2 \otimes \dots \otimes x_{14}$ with the multiple contraction $\mu_{(ci)(dk)(el)(fn)(gm)(hj)}$, for example, is given by the replacement rule

```

ts[a_, b_, c_, d_, e_, f_, g_, h_, i_, j_, k_, l_, m_, n_] :=
mu[c, i] mu[d, k] mu[e, l] mu[f, n] mu[g, m] mu[h, j]

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where μ is the function which computes the value of the bilinear form μ .

5. FINAL REMARK

While an $\mathrm{Sp}(2g, \mathbb{Q})$ -invariant map $\mathfrak{h}_{g,1}(12) \rightarrow \mathbb{Q}$ inducing the isomorphism $H_1(\mathfrak{h}_{g,1}^+)_{12}^{\mathrm{Sp}} \cong \mathbb{Q}$ is unique up to scalar for $g \geq 8$, a description of this map using multiple contractions such as our cocycle C is not so. It might be possible to obtain more simple expression than C by taking another set of multiple contractions as a coordinate system of $\mathfrak{h}_{g,1}(12)^{\mathrm{Sp}} \cong \mathbb{Q}^{650}$.

In the study of the structure of the Lie algebra $\mathfrak{h}_{g,1}^+$, the determination of the Lie subalgebra

$$J = \bigoplus_{k=1}^{\infty} J(k), \quad J(k) \subset \mathfrak{h}_{g,1}(k)$$

of $\mathfrak{h}_{g,1}^+$ generated by the degree 1 part $\mathfrak{h}_{g,1}(1)$ has been considered to be important. It was shown by Hain [10] that this problem is the same as the determination of the rational image of the Johnson homomorphisms for subgroups of the mapping class groups of a surface. To this problem, Enomoto and Satoh [8] provided the following powerful tool. They showed that the $\mathrm{Sp}(2g, \mathbb{Q})$ -equivariant map obtained as the composition

$$ES_k : \mathfrak{h}_{g,1}(k) \hookrightarrow H \otimes \mathcal{L}_{k+1}(H) \hookrightarrow H^{\otimes(k+2)} \xrightarrow{\mu \otimes (\mathrm{id}^{\otimes k})} H^{\otimes k} \rightarrow (H^{\otimes k})_{\mathbb{Z}/k\mathbb{Z}}$$

is not trivial in general, but its restriction to $J(k)$ is trivial for any $k \geq 2$. Here $(H^{\otimes k})_{\mathbb{Z}/k\mathbb{Z}}$ denotes the coinvariant quotient of $H^{\otimes k}$ with respect to the action of $\mathbb{Z}/k\mathbb{Z}$ as rotations of the entries. Consequently, the image of ES_k gives a lower-bound estimate of the gap between J_k and $\mathfrak{h}_{g,1}(k)$. There are also important papers of Conant [3] and Conant-Kassabov [4] on this subject.

Using some data and Mathematica codes for the computation of the previous sections, the authors observed the following relationship between ES_{12} and our cocycle C .

Theorem 5.1. *For $g \geq 6$, the cocycle $C : \mathfrak{h}_{g,1}(12) \rightarrow \mathbb{Q}$ factors through $\mathrm{Im} ES_{12}$.*

Sketch of Proof. First we see that the image of $ES_{12} : \mathfrak{h}_{g,1}(12)^{\mathrm{Sp}} \rightarrow (H^{\otimes 12})_{\mathbb{Z}/12\mathbb{Z}}^{\mathrm{Sp}} \cong \mathbb{Q}^{897}$ is 284-dimensional and take a basis of $\mathrm{Ker} ES_{12}$. Then we observe that the map C is trivial on the $\mathrm{Ker} ES_{12}$ by applying C to the basis. \square

By using this theorem, it is possible to give another description of the $\mathrm{Sp}(2g, \mathbb{Q})$ -invariant map $C : \mathfrak{h}_{g,1}(12) \rightarrow \mathbb{Q}$ in the form $C = C' \circ ES_{12}$ with an $\mathrm{Sp}(2g, \mathbb{Q})$ -invariant map $C' : \mathrm{Im} ES_{12} \subset (H^{\otimes 12})_{\mathbb{Z}/12\mathbb{Z}}^{\mathrm{Sp}} \rightarrow \mathbb{Q}$. Such a map C' is described by chord diagrams with 6 chords serving as a coordinate system of $(H^{\otimes 12})_{\mathbb{Z}/12\mathbb{Z}}^{\mathrm{Sp}} \cong \mathbb{Q}^{897}$. The details of the above computation will appear elsewhere.

APPENDIX A. A COORDINATE SYSTEM OF $\mathfrak{h}_{\infty,1}(12)^{\text{Sp}} \cong \mathbb{Q}^{650}$

The authors found the following 650 multiple contractions

$$(C_1, \dots, C_{650}) : \mathfrak{h}_{g,1}(12)^{\text{Sp}} \longrightarrow \mathbb{Q}^{650}$$

forming a coordinate system of $\mathfrak{h}_{g,1}(12)^{\text{Sp}}$ for $g \geq 5$ (see [20, Theorem 1.2]). The multiple contraction C_m for $1 \leq m \leq 650$ is given below in the form

$$m \ (i_2 j_2)(i_3 j_3)(i_4 j_4)(i_5 j_5)(i_6 j_6)(i_7 j_7),$$

which means the restriction of the $\text{Sp}(2g, \mathbb{Q})$ -invariant map

$$\mu_{(ab)(i_2 j_2)(i_3 j_3) \dots (i_7 j_7)} : H^{\otimes 14} \longrightarrow \mathbb{Q}$$

to the subspace $\mathfrak{h}_{g,1}(12)$.

1 (ci)(dk)(el)(fn)(gm)(hj)	2 (ci)(dk)(em)(fl)(gn)(hj)	3 (ci)(dk)(el)(fm)(gn)(hj)	4 (ci)(dj)(en)(fm)(gl)(hk)
5 (ci)(dj)(em)(fn)(gl)(hk)	6 (ci)(dj)(en)(fl)(gm)(hk)	7 (ci)(dj)(el)(fn)(gm)(hk)	8 (ci)(dj)(em)(fl)(gn)(hk)
9 (ci)(dj)(el)(fm)(gn)(hk)	10 (ci)(dj)(en)(fm)(gk)(hl)	11 (ci)(dj)(em)(fn)(gk)(hl)	12 (ci)(dj)(en)(fk)(gm)(hl)
13 (ci)(dk)(ej)(fn)(gm)(hl)	14 (ci)(dj)(ek)(fn)(gm)(hl)	15 (ci)(dk)(em)(fj)(gn)(hl)	16 (ci)(dj)(em)(fk)(gn)(hl)
17 (ci)(dk)(ej)(fm)(gn)(hl)	18 (ci)(dj)(ek)(fm)(gn)(hl)	19 (ci)(dk)(el)(fn)(gj)(hm)	20 (ci)(dj)(en)(fl)(gk)(hm)
21 (ci)(dj)(el)(fn)(gk)(hm)	22 (ci)(dj)(en)(fk)(gl)(hm)	23 (ci)(dk)(ej)(fn)(gl)(hm)	24 (ci)(dj)(ek)(fn)(gl)(hm)
25 (ci)(dk)(el)(fj)(gn)(hm)	26 (ci)(dj)(el)(fk)(gn)(hm)	27 (ci)(dk)(ej)(fl)(gn)(hm)	28 (ci)(dj)(ek)(fl)(gn)(hm)
29 (ci)(dk)(em)(fl)(gj)(hn)	30 (ci)(dj)(em)(fl)(gk)(hn)	31 (ci)(dj)(el)(fm)(gk)(hn)	32 (ci)(dk)(em)(fj)(gl)(hn)
33 (ci)(dj)(em)(fk)(gl)(hn)	34 (ci)(dj)(ek)(fm)(gl)(hn)	35 (cd)(el)(fn)(gm)(hk)(ij)	36 (cd)(en)(fk)(gm)(hl)(ij)
37 (cd)(ek)(fn)(gm)(hl)(ij)	38 (cd)(ek)(fm)(gn)(hl)(ij)	39 (cd)(el)(fn)(gk)(hm)(ij)	40 (cd)(ek)(fn)(gl)(hm)(ij)
41 (cd)(ek)(fl)(gn)(hm)(ij)	42 (cd)(el)(fn)(gm)(hj)(ik)	43 (cd)(en)(fm)(gj)(hl)(ik)	44 (cd)(em)(fn)(gj)(hl)(ik)
45 (cd)(en)(fj)(gm)(hl)(ik)	46 (ce)(dj)(fn)(gm)(hl)(ik)	47 (cd)(ej)(fn)(gm)(hl)(ik)	48 (ce)(dj)(fn)(gl)(hm)(ik)
49 (cd)(ej)(fn)(gl)(hm)(ik)	50 (cd)(el)(fj)(gn)(hm)(ik)	51 (cd)(en)(fj)(gl)(hm)(ik)	52 (ce)(dj)(fn)(gl)(hm)(ik)
53 (cd)(ej)(fn)(gl)(hm)(ik)	54 (cd)(el)(fj)(gn)(hm)(ik)	55 (ce)(dj)(fl)(gn)(hm)(ik)	56 (cd)(ej)(fl)(gn)(hm)(ik)
57 (ce)(dj)(fm)(gl)(hn)(ik)	58 (cd)(ej)(fm)(gl)(hn)(ik)	59 (ce)(dj)(fl)(gm)(hn)(ik)	60 (cd)(ej)(fl)(gm)(hn)(ik)
61 (cd)(en)(fk)(gm)(hj)(il)	62 (cd)(ek)(fn)(gm)(hj)(il)	63 (cd)(ek)(fm)(gn)(hj)(il)	64 (cd)(en)(fj)(gm)(hk)(il)
65 (ce)(dj)(fn)(gm)(hk)(il)	66 (cd)(ej)(fn)(gm)(hk)(il)	67 (ce)(dj)(fm)(gn)(hk)(il)	68 (cd)(ej)(fm)(gn)(hk)(il)
69 (cd)(ek)(fn)(gj)(hm)(il)	70 (cd)(en)(fj)(gk)(hm)(il)	71 (ce)(dj)(fn)(gk)(hm)(il)	72 (cd)(ej)(fn)(gk)(hm)(il)
73 (cd)(ek)(fj)(gn)(hm)(il)	74 (ce)(dj)(fk)(gn)(hm)(il)	75 (cd)(ej)(fk)(gn)(hm)(il)	76 (ce)(dj)(fm)(gk)(hn)(il)
77 (cd)(ej)(fm)(gk)(hn)(il)	78 (cd)(ek)(fj)(gm)(hn)(il)	79 (ce)(dj)(fk)(gm)(hn)(il)	80 (cd)(ej)(fk)(gm)(hn)(il)
81 (cd)(el)(fn)(gk)(hj)(im)	82 (cd)(ek)(fn)(gl)(hj)(im)	83 (cd)(ek)(fl)(gn)(hj)(im)	84 (cd)(el)(fn)(gj)(hk)(im)
85 (ce)(dj)(fn)(gl)(hk)(im)	86 (cd)(ej)(fn)(gl)(hk)(im)	87 (ce)(dj)(fl)(gn)(hk)(im)	88 (cd)(ej)(fl)(gn)(hk)(im)
89 (cd)(ek)(fn)(gj)(hl)(im)	90 (ce)(dj)(fn)(gk)(hl)(im)	91 (cd)(ek)(fj)(gn)(hl)(im)	92 (cd)(ek)(fj)(gn)(hl)(im)
93 (ce)(dj)(fk)(gn)(hl)(im)	94 (cd)(ej)(fk)(gn)(hl)(im)	95 (ce)(dj)(fl)(gk)(hn)(im)	96 (cd)(ej)(fl)(gk)(hn)(im)
97 (ce)(dj)(fk)(gl)(hn)(im)	98 (cd)(ej)(fk)(gl)(hn)(im)	99 (ce)(dj)(fm)(gl)(hk)(in)	100 (ce)(dj)(fl)(gm)(hk)(in)
101 (ce)(dj)(fm)(gk)(hl)(in)	102 (ce)(dj)(fk)(gm)(hl)(in)	103 (cd)(ej)(fk)(gm)(hl)(in)	104 (ce)(dj)(fl)(gk)(hm)(in)
105 (ce)(dj)(fk)(gl)(hm)(in)	106 (ce)(di)(fn)(gm)(hl)(jk)	107 (cd)(ei)(fn)(gm)(hl)(jk)	108 (ce)(di)(fm)(gn)(hl)(jk)
109 (cd)(ei)(fm)(gn)(hl)(jk)	110 (ce)(di)(fn)(gl)(hm)(jk)	111 (cd)(ei)(fn)(gl)(hm)(jk)	112 (ce)(di)(fl)(gn)(hm)(jk)
113 (cd)(ei)(fl)(gn)(hm)(jk)	114 (ce)(di)(fm)(gl)(hn)(jk)	115 (cd)(ei)(fm)(gl)(hn)(jk)	116 (ce)(di)(fl)(gm)(hn)(jk)
117 (cd)(ei)(fl)(gm)(hn)(jk)	118 (cd)(en)(fm)(gh)(il)(jk)	119 (cd)(em)(fn)(gh)(il)(jk)	120 (cf)(dh)(en)(gm)(il)(jk)
121 (cd)(en)(fh)(gm)(il)(jk)	122 (ce)(dh)(fn)(gm)(il)(jk)	123 (cd)(eh)(fn)(gm)(il)(jk)	124 (cf)(dh)(em)(gn)(il)(jk)
125 (ce)(dh)(fm)(gn)(il)(jk)	126 (cd)(eh)(fm)(gn)(il)(jk)	127 (cf)(dg)(en)(hm)(il)(jk)	128 (cd)(en)(fg)(hm)(il)(jk)
129 (ce)(dg)(fn)(hm)(il)(jk)	130 (cd)(eg)(fn)(hm)(il)(jk)	131 (ce)(df)(gn)(hm)(il)(jk)	132 (cf)(dg)(em)(hn)(il)(jk)
133 (ce)(dg)(fm)(hn)(il)(jk)	134 (cd)(eg)(fm)(hn)(il)(jk)	135 (ce)(df)(gm)(hn)(il)(jk)	136 (cd)(el)(fn)(gh)(im)(jk)
137 (cf)(dh)(en)(gl)(im)(jk)	138 (cd)(en)(fh)(gl)(im)(jk)	139 (ce)(dh)(fn)(gl)(im)(jk)	140 (cd)(eh)(fn)(gl)(im)(jk)
141 (cf)(dh)(el)(gn)(im)(jk)	142 (cd)(el)(fh)(gn)(im)(jk)	143 (ce)(dh)(fl)(gn)(im)(jk)	144 (cd)(eh)(fl)(gn)(im)(jk)
145 (cf)(dg)(en)(hl)(im)(jk)	146 (cd)(en)(fg)(hl)(im)(jk)	147 (ce)(dg)(fn)(hl)(im)(jk)	148 (cd)(eg)(fn)(hl)(im)(jk)
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157 (cd)(eh)(fm)(gl)(in)(jk)	158 (cf)(dh)(el)(gm)(in)(jk)	159 (ce)(dh)(fl)(gm)(in)(jk)	160 (cd)(eh)(fl)(gm)(in)(jk)
161 (cf)(dg)(em)(hl)(in)(jk)	162 (ce)(dg)(fm)(hl)(in)(jk)	163 (cd)(eg)(fm)(hl)(in)(jk)	164 (ce)(df)(gm)(hl)(in)(jk)
165 (cf)(dl)(eg)(hm)(in)(jk)	166 (cf)(dg)(el)(hm)(in)(jk)	167 (ce)(dg)(fl)(hm)(in)(jk)	168 (cd)(eg)(fl)(hm)(in)(jk)
169 (ce)(df)(gl)(hm)(in)(jk)	170 (cd)(en)(fi)(gm)(hk)(jl)	171 (ce)(di)(fn)(gm)(hk)(jl)	172 (cd)(ei)(fn)(gm)(hk)(jl)

- 173 $(ce)(di)(fm)(gn)(hk)(jl)$ 174 $(cd)(ei)(fm)(gn)(hk)(jl)$ 175 $(cd)(en)(fi)(gk)(hm)(jl)$ 176 $(ce)(di)(fn)(gk)(hm)(jl)$
 177 $(cd)(ei)(fn)(gk)(hm)(jl)$ 178 $(ce)(di)(fk)(gn)(hm)(jl)$ 179 $(cd)(ei)(fk)(gn)(hm)(jl)$ 180 $(ce)(di)(fm)(gk)(hn)(jl)$
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413 (cf)(dh)(ei)(gn)(jm)(kl)	414 (ce)(di)(fh)(gn)(jm)(kl)	415 (cd)(ei)(fh)(gn)(jm)(kl)	416 (ce)(dh)(fi)(gn)(jm)(kl)
417 (cd)(eh)(fi)(gn)(jm)(kl)	418 (ce)(dg)(fn)(hi)(jm)(kl)	419 (cd)(eg)(fn)(hi)(jm)(kl)	420 (ce)(df)(gn)(hi)(jm)(kl)
421 (cd)(ef)(gn)(hi)(jm)(kl)	422 (cf)(di)(eg)(hn)(jm)(kl)	423 (cf)(dg)(ei)(hn)(jm)(kl)	424 (ce)(dg)(fi)(hn)(jm)(kl)
425 (cd)(eg)(fi)(hn)(jm)(kl)	426 (ce)(df)(gi)(hn)(jm)(kl)	427 (cd)(ef)(gi)(hn)(jm)(kl)	428 (cf)(dh)(eg)(in)(jm)(kl)
429 (cf)(dg)(eh)(in)(jm)(kl)	430 (cd)(eh)(fg)(in)(jm)(kl)	431 (cd)(eg)(fh)(in)(jm)(kl)	432 (ce)(dh)(fm)(gi)(jn)(kl)
433 (cd)(eh)(fm)(gi)(jn)(kl)	434 (cf)(di)(eh)(gm)(jn)(kl)	435 (cf)(dh)(ei)(gm)(jn)(kl)	436 (ce)(di)(fh)(gm)(jn)(kl)
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445 (cd)(ei)(fg)(hm)(jn)(kl)	446 (ce)(dg)(fi)(hm)(jn)(kl)	447 (cd)(eg)(fi)(hm)(jn)(kl)	448 (ce)(df)(gi)(hm)(jn)(kl)
449 (cf)(dg)(gi)(hm)(jn)(kl)	450 (cf)(dh)(eg)(im)(jn)(kl)	451 (cf)(dg)(eh)(im)(jn)(kl)	452 (cd)(eh)(fi)(im)(jn)(kl)
453 (cd)(eg)(fh)(im)(jn)(kl)	454 (cd)(ej)(fn)(gl)(hi)(km)	455 (cd)(ej)(fl)(gn)(hi)(km)	456 (cd)(el)(fn)(gi)(hj)(km)
457 (cd)(ei)(fn)(gl)(hj)(km)	458 (cd)(ei)(fl)(gn)(hj)(km)	459 (cd)(ej)(fn)(gi)(hl)(km)	460 (cd)(ei)(fn)(gj)(hl)(km)
461 (cf)(dj)(ei)(gn)(hl)(km)	462 (cf)(di)(ej)(gn)(hl)(km)	463 (cd)(ej)(fi)(gn)(hl)(km)	464 (ce)(di)(fj)(gn)(hl)(km)
465 (cd)(ei)(fj)(gn)(hl)(km)	466 (cd)(ej)(fl)(gi)(hn)(km)	467 (cd)(ei)(fl)(gj)(hn)(km)	468 (cf)(dj)(ei)(gl)(hn)(km)
469 (cf)(di)(ej)(gl)(hn)(km)	470 (cd)(ej)(fi)(gl)(hn)(km)	471 (ce)(di)(fj)(gl)(hn)(km)	472 (cd)(ei)(fj)(gl)(hn)(km)
473 (cd)(eg)(fl)(hn)(ij)(km)	474 (cd)(ef)(gl)(hn)(ij)(km)	475 (cg)(dh)(ej)(fn)(il)(km)	476 (cd)(eh)(fn)(gj)(il)(km)
477 (cf)(dj)(eh)(gn)(il)(km)	478 (cf)(dh)(ej)(gn)(il)(km)	479 (ce)(dh)(fj)(gn)(il)(km)	480 (cd)(eh)(fj)(gn)(il)(km)
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537 (cd)(eh)(fg)(il)(jn)(km)	538 (ce)(dg)(fh)(il)(jn)(km)	539 (cd)(eg)(fh)(il)(jn)(km)	540 (ce)(df)(gh)(il)(jn)(km)
541 (cd)(ef)(fh)(il)(jn)(km)	542 (cf)(di)(ej)(gm)(hl)(kn)	543 (ce)(di)(fj)(gm)(hl)(kn)	544 (ce)(di)(fj)(gl)(hm)(kn)
545 (cd)(eg)(fl)(hm)(ij)(kn)	546 (cf)(dh)(ej)(gm)(il)(kn)	547 (ce)(dh)(fj)(gm)(il)(kn)	548 (cd)(eh)(fj)(gm)(il)(kn)
549 (ce)(dg)(fm)(hj)(il)(kn)	550 (cd)(eg)(fm)(hj)(il)(kn)	551 (ce)(df)(gm)(hj)(il)(kn)	552 (cf)(dg)(ej)(hm)(il)(kn)
553 (ce)(dg)(fj)(hm)(il)(kn)	554 (cd)(eg)(fj)(hm)(il)(kn)	555 (ce)(df)(gj)(hm)(il)(kn)	556 (ce)(dh)(fl)(gj)(im)(kn)
557 (ce)(dh)(fj)(gl)(im)(kn)	558 (cd)(eg)(fl)(hj)(im)(kn)	559 (cf)(dg)(ej)(hl)(im)(kn)	560 (ce)(dg)(fj)(hl)(im)(kn)
561 (cd)(eg)(fj)(hl)(im)(kn)	562 (ce)(df)(gj)(hl)(im)(kn)	563 (ce)(dh)(fm)(gi)(jl)(kn)	564 (cf)(di)(eh)(gm)(jl)(kn)
565 (cf)(dh)(ei)(gm)(jl)(kn)	566 (ce)(di)(fh)(gm)(jl)(kn)	567 (ce)(dh)(fi)(gm)(jl)(kn)	568 (cd)(eh)(fi)(gm)(jl)(kn)
569 (cf)(dg)(ei)(hm)(jl)(kn)	570 (cd)(eg)(fi)(hm)(jl)(kn)	571 (ce)(df)(gi)(hm)(jl)(kn)	572 (cd)(ef)(gi)(hm)(jl)(kn)
573 (cf)(dg)(eh)(im)(jl)(kn)	574 (cd)(eh)(fg)(im)(jl)(kn)	575 (cd)(eg)(fh)(im)(jl)(kn)	576 (ce)(dh)(fl)(gi)(jm)(kn)
577 (ce)(di)(fh)(gl)(jm)(kn)	578 (ce)(dh)(fi)(gl)(jm)(kn)	579 (ce)(df)(gl)(hi)(jm)(kn)	580 (ce)(dg)(fi)(hl)(jm)(kn)
581 (cd)(eg)(fi)(hl)(jm)(kn)	582 (ce)(df)(gi)(hl)(jm)(kn)	583 (cd)(ef)(gi)(hl)(jm)(kn)	584 (cd)(eh)(fg)(il)(jm)(kn)
585 (cd)(eg)(fh)(il)(jm)(kn)	586 (cd)(ej)(fn)(gk)(hi)(lm)	587 (cd)(ej)(fk)(gn)(hi)(lm)	588 (cd)(ei)(fn)(gk)(hj)(lm)
589 (cd)(ei)(fk)(gn)(hj)(lm)	590 (cd)(ej)(fn)(gi)(hk)(lm)	591 (cd)(ei)(fn)(gj)(hk)(lm)	592 (cd)(ej)(fi)(gn)(hk)(lm)
593 (cd)(ei)(fj)(gn)(hk)(lm)	594 (cd)(ej)(fk)(gi)(hn)(lm)	595 (cd)(ei)(fk)(gj)(hn)(lm)	596 (cd)(ej)(fi)(gk)(hn)(lm)
597 (cd)(ei)(fj)(gk)(hn)(lm)	598 (cd)(eg)(fn)(hk)(ij)(lm)	599 (cd)(eg)(fk)(hn)(ij)(lm)	600 (cd)(ef)(gk)(hn)(ij)(lm)
601 (cd)(eh)(fn)(gj)(ik)(lm)	602 (cd)(eh)(fj)(gn)(ik)(lm)	603 (cd)(eg)(fn)(hj)(ik)(lm)	604 (ce)(dg)(fj)(hn)(ik)(lm)
605 (cd)(eg)(fj)(hn)(ik)(lm)	606 (ce)(df)(gj)(hn)(ik)(lm)	607 (cd)(ef)(gj)(hn)(ik)(lm)	608 (cd)(eh)(fk)(gj)(in)(lm)
609 (ce)(dh)(fj)(gk)(in)(lm)	610 (cd)(eh)(fj)(gk)(in)(lm)	611 (cd)(eg)(fk)(hj)(in)(lm)	612 (cd)(ef)(gk)(hj)(in)(lm)
613 (ce)(dg)(fj)(hk)(in)(lm)	614 (cd)(eg)(fj)(hk)(in)(lm)	615 (ce)(df)(gj)(hk)(in)(lm)	616 (cd)(ef)(gj)(hk)(in)(lm)
617 (cd)(eg)(fi)(hn)(jk)(lm)	618 (cd)(eg)(fh)(in)(jk)(lm)	619 (cd)(eh)(fk)(gi)(jn)(lm)	620 (ce)(dh)(fi)(gk)(jn)(lm)
621 (cd)(eh)(fi)(gk)(jn)(lm)	622 (cd)(eg)(fk)(hi)(jn)(lm)	623 (ce)(dg)(fi)(hk)(jn)(lm)	624 (cd)(eg)(fi)(hk)(jn)(lm)
625 (ce)(df)(gi)(hk)(jn)(lm)	626 (cd)(ef)(gi)(hk)(jn)(lm)	627 (cd)(eg)(fh)(ik)(jn)(lm)	628 (cd)(ef)(gh)(ik)(jn)(lm)
629 (cd)(eg)(fj)(hi)(kn)(lm)	630 (ce)(df)(gj)(hi)(kn)(lm)	631 (ce)(dg)(fj)(hj)(kn)(lm)	632 (cd)(eg)(fi)(hj)(kn)(lm)
633 (ce)(df)(gi)(hj)(kn)(lm)	634 (cd)(ef)(gi)(hj)(kn)(lm)	635 (cd)(eg)(fh)(ij)(kn)(lm)	636 (ce)(di)(fm)(gj)(hk)(ln)
637 (ce)(di)(fj)(gm)(hk)(ln)	638 (ce)(dh)(fk)(gm)(ij)(ln)	639 (ce)(dg)(fj)(hm)(ik)(ln)	640 (cd)(eg)(fj)(hm)(ik)(ln)
641 (ce)(df)(gj)(hm)(ik)(ln)	642 (ce)(dg)(fj)(hk)(im)(ln)	643 (ce)(df)(gj)(hk)(im)(ln)	644 (cd)(eg)(fi)(hm)(jk)(ln)
645 (cd)(eg)(fh)(im)(jk)(ln)	646 (ce)(df)(gi)(hk)(jm)(ln)	647 (cd)(eg)(fh)(ik)(jm)(ln)	648 (cf)(dg)(en)(hl)(ik)(jm)
649 (cg)(dh)(en)(fi)(jl)(km)	650 (ci)(dk)(em)(fn)(gl)(hj)		

APPENDIX B. THE IMAGE OF THE BRACKET MAP

The brackets of the following 649 elements generate a 649-dimensional subspace W of $\mathfrak{h}_{g,1}(12)^{\text{Sp}} \cong \mathbb{Q}^{650}$ for $g \geq 8$. In the computation, three types of bracket maps are used, where Type 1 and Type 2 concern with $[\mathfrak{h}_{g,1}(1), \mathfrak{h}_{g,1}(11)]$ and Type 3 with $[\mathfrak{h}_{g,1}(3), \mathfrak{h}_{g,1}(9)]$.

Type 1 1-469 : $[S(a_1, b_1, a_8), S(b_8, X)] = S(a_1, b_1, X)$, where X is a sequence obtained by permuting $\{a_2, b_2, \dots, a_7, b_7\}$ as follows:

1-102

[illegible]

[illegible]

[illegible]

[illegible]

Type 2 470–648 : $[S(a_1, a_2, a_3), S(b_3, Y, b_1)]$, where Y is a word obtained by permuting $\{b_2, a_4, b_4, a_5, b_5, \dots, a_8, b_8\}$ as follows:

470-592

[illegible]

593–648

$\{a_4, a_5, b_4, a_6, a_7, b_6, a_8, b_2, b_5, b_8, b_7\}$, $\{a_4, a_5, b_4, a_6, a_7, b_6, b_2, a_8, b_5, b_8, b_7\}$, $\{a_4, a_5, b_4, a_6, b_2, b_6, a_7, a_8, b_5, b_7, b_8\}$,
 $\{a_4, a_5, b_4, a_6, b_2, b_6, a_7, a_8, b_5, b_8, b_7\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_6, b_7, b_5, b_8, b_2\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_6, b_8, b_5, b_7, b_2\}$,
 $\{a_4, a_5, b_4, a_6, a_7, a_8, b_6, b_2, b_5, b_7, b_8\}$, $\{a_4, a_5, b_4, a_6, a_7, b_2, b_6, a_8, b_5, b_7, b_8\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_6, b_2, b_5, b_8, b_7\}$,
 $\{a_4, a_5, b_4, a_6, a_7, b_2, b_6, a_8, b_5, b_8, b_7\}$, $\{a_4, a_5, b_4, a_6, b_2, a_7, b_6, a_8, b_5, b_7, b_8\}$, $\{a_4, a_5, b_4, a_6, b_2, a_7, b_6, a_8, b_5, b_8, b_7\}$,
 $\{a_4, a_5, b_4, a_6, a_7, a_8, b_7, b_6, b_5, b_8, b_2\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_7, b_8, b_5, b_6, b_2\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_7, b_2, b_5, b_6, b_8\}$,
 $\{a_4, a_5, b_4, a_6, a_7, a_8, b_7, b_2, b_5, b_8, b_6\}$, $\{a_4, a_5, b_4, a_6, a_7, b_2, b_7, a_8, b_5, b_8, b_6\}$, $\{a_4, a_5, b_4, a_6, a_7, b_6, b_7, a_8, b_2, b_5, b_8\}$,
 $\{a_4, a_5, b_4, a_6, a_7, b_6, a_8, b_7, b_8, b_5, b_2\}$, $\{a_4, a_5, b_4, a_6, a_7, b_6, a_8, b_7, b_2, b_5, b_8\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_6, b_7, b_8, b_5, b_2\}$,
 $\{a_4, a_5, b_4, a_6, a_7, a_8, b_6, b_7, b_2, b_5, b_8\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_6, b_8, b_7, b_5, b_2\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_6, b_8, b_2, b_5, b_7\}$,
 $\{a_4, a_5, b_4, a_6, a_7, a_8, b_7, b_6, b_2, b_5, b_8\}$, $\{a_4, a_5, b_4, a_6, a_7, a_8, b_7, b_8, b_6, b_5, b_2\}$, $\{a_4, a_5, a_6, b_4, b_5, b_6, a_7, a_8, b_2, b_7, b_8\}$,
 $\{a_4, a_5, a_6, b_4, b_5, a_7, b_6, a_8, b_2, b_7, b_8\}$, $\{a_4, a_5, a_6, b_4, b_5, a_7, a_8, b_6, b_8, b_7, b_2\}$, $\{a_4, a_5, a_6, b_4, b_5, a_7, a_8, b_6, b_2, b_7, b_8\}$,
 $\{a_4, a_5, a_6, b_4, b_5, a_7, a_8, b_7, b_6, b_8, b_2\}$, $\{a_4, a_5, a_6, b_4, b_5, a_7, a_8, b_7, b_8, b_6, b_2\}$, $\{a_4, a_5, a_6, b_4, b_5, a_7, b_2, a_8, b_6, b_7, b_8\}$,
 $\{a_4, a_5, a_6, b_4, b_5, a_7, a_8, b_2, b_6, b_8, b_7\}$, $\{a_4, a_5, a_6, b_4, b_5, a_7, a_8, b_7, b_8, b_6, b_2\}$, $\{a_4, a_5, b_2, b_4, b_5, a_6, a_7, a_8, b_7, b_6, b_8\}$,
 $\{a_4, a_5, a_6, b_4, a_7, b_5, b_6, a_8, b_2, b_7, b_8\}$, $\{a_4, a_5, a_6, b_4, a_7, b_5, a_8, b_6, b_8, b_7, b_2\}$, $\{a_4, a_5, a_6, b_4, a_7, b_5, a_8, b_6, b_2, b_7, b_8\}$,
 $\{a_4, a_5, a_6, b_4, a_7, b_5, a_8, b_7, b_6, b_8, b_2\}$, $\{a_4, a_5, a_6, b_4, a_7, b_5, a_8, b_2, b_6, b_7, b_8\}$, $\{a_4, a_5, a_6, b_4, a_7, b_5, a_8, b_2, b_6, b_7, b_8\}$,
 $\{a_4, a_5, a_6, b_4, a_7, a_8, b_5, b_6, b_7, b_8, b_2\}$, $\{a_4, a_5, a_6, b_4, a_7, a_8, b_5, b_6, b_8, b_7, b_2\}$, $\{a_4, a_5, a_6, b_4, a_7, a_8, b_5, b_7, b_2, b_6, b_8\}$,
 $\{a_4, a_5, a_6, b_4, a_7, a_8, b_5, b_8, b_7, b_6, b_2\}$, $\{a_4, a_5, a_6, b_4, a_7, a_8, b_2, b_5, b_6, b_7, b_8\}$, $\{a_4, a_5, a_6, b_4, a_7, b_2, a_8, b_5, b_6, b_7, b_8\}$,
 $\{a_4, a_5, b_2, b_4, a_6, a_7, a_8, b_5, b_6, b_7, b_8\}$, $\{a_4, a_5, a_6, b_5, a_7, a_8, b_6, b_7, b_8, b_4, b_2\}$

Type 3 649 : $[S(b_1, a_4, a_3, a_1, a_2), S(b_2, a_5, a_6, b_5, a_7, b_6, a_8, b_3, b_7, b_8, b_4)]$

APPENDIX C. THE COCYCLE C

The cocycle C is given as a linear combination of C_1, C_2, \dots, C_{647} in Appendix A. The explicit formula is as follows.

$$\begin{aligned}
C = & 2160C_1 - 2616C_2 - 180C_3 - 384C_4 - 1956C_5 - 528C_6 + 1596C_7 - 2100C_8 \\
& + 1524C_9 - 240C_{10} - 1812C_{11} - 1020C_{12} + 132C_{13} + 1344C_{14} + 1584C_{15} - 2520C_{16} \\
& + 2400C_{17} - 156C_{18} - 1224C_{19} - 1020C_{20} - 4236C_{21} - 1164C_{22} - 12C_{23} - 4140C_{24} \\
& - 1224C_{25} - 7140C_{26} - 1032C_{27} - 7428C_{28} - 840C_{29} - 2736C_{30} - 3228C_{31} - 936C_{32} \\
& - 2736C_{33} - 4680C_{34} + 496C_{35} + 524C_{36} + 3780C_{37} + 4028C_{38} - 788C_{39} + 1264C_{40} \\
& - 3384C_{41} - 1076C_{42} + 516C_{43} + 372C_{44} - 1664C_{45} + 3000C_{46} + 1980C_{47} + 756C_{48} \\
& + 4672C_{49} - 5768C_{50} - 3204C_{51} + 3072C_{52} - 1540C_{53} - 6316C_{54} - 6268C_{55} - 9808C_{56} \\
& - 3476C_{57} - 7832C_{58} - 8640C_{59} - 13092C_{60} + 4516C_{61} + 5060C_{62} + 8092C_{63} - 848C_{64} \\
& + 2976C_{65} + 1992C_{66} + 588C_{67} + 772C_{68} + 1088C_{69} + 244C_{70} + 5824C_{71} + 4860C_{72} \\
& + 2136C_{73} + 2280C_{74} - 1296C_{75} + 664C_{76} - 3260C_{77} - 4436C_{78} + 916C_{79} - 7724C_{80} \\
& - 2296C_{81} + 6256C_{82} + 9744C_{83} - 1780C_{84} + 4808C_{85} + 4796C_{86} - 700C_{87} + 1284C_{88} \\
& + 3784C_{89} + 5580C_{90} + 7172C_{91} + 4832C_{92} + 4832C_{93} + 5168C_{94} + 84C_{95} + 564C_{96} \\
& + 2088C_{97} + 1980C_{98} - 804C_{99} - 1756C_{100} - 692C_{101} + 2928C_{102} - 2496C_{103} - 1344C_{104} \\
& + 516C_{105} + 2532C_{106} - 78876C_{107} + 3684C_{108} - 234400C_{109} + 3612C_{110} - 49828C_{111} - 4452C_{112} \\
& - 122892C_{113} + 696C_{114} - 122436C_{115} - 10032C_{116} + 10844C_{117} - 26378C_{118} - 20362C_{119} + 924C_{120} \\
& - 37936C_{121} + 2172C_{122} - 179514C_{123} + 792C_{124} + 3792C_{125} - 409286C_{126} + 1740C_{127} - 215036C_{128} \\
& - 9924C_{129} - 499622C_{130} + 20316C_{131} - 1920C_{132} + 27168C_{133} - 553120C_{134} + 1812C_{135} - 8440C_{136} \\
& - 3996C_{137} - 11222C_{138} - 9432C_{139} - 114290C_{140} + 960C_{141} - 15018C_{142} - 4944C_{143} - 349734C_{144} \\
& - 8724C_{145} - 196928C_{146} - 9636C_{147} - 368506C_{148} + 16476C_{149} + 22116C_{150} + 14304C_{151} - 624962C_{152} \\
& + 7548C_{153} + 81792C_{154} - 1716C_{155} + 19188C_{156} - 124728C_{157} + 20304C_{158} + 8712C_{159} + 8948C_{160} \\
& - 7248C_{161} + 7080C_{162} - 495912C_{163} + 2964C_{164} + 3060C_{165} + 14928C_{166} + 204C_{167} - 392446C_{168} \\
& - 11388C_{169} - 23126C_{170} + 2532C_{171} - 97280C_{172} + 4656C_{173} - 201028C_{174} - 43634C_{175} + 1056C_{176} \\
& - 83688C_{177} - 5844C_{178} - 188572C_{179} + 1752C_{180} - 112556C_{181} + 44288C_{182} - 10416C_{183} + 5100C_{184} \\
& + 288C_{185} + 288C_{186} - 153500C_{187} - 188918C_{188} + 8916C_{189} - 19486C_{190} + 17424C_{191} - 119176C_{192} \\
& + 8340C_{193} + 18612C_{194} - 390946C_{195} + 17580C_{196} - 138656C_{197} + 13176C_{198} - 388138C_{199} + 43860C_{200} \\
& + 6216C_{201} + 42864C_{202} - 412368C_{203} + 25356C_{204} + 34612C_{205} + 4500C_{206} - 25844C_{207} - 6192C_{208} \\
& - 154118C_{209} - 10992C_{210} - 29418C_{211} - 15360C_{212} - 297790C_{213} + 11472C_{214} - 21748C_{215} + 13368C_{216} \\
& - 145026C_{217} + 13428C_{218} + 28258C_{219} - 8664C_{220} + 130460C_{221} - 25164C_{222} - 428252C_{223} - 20232C_{224} \\
& + 440158C_{225} - 204C_{226} + 18192C_{227} - 118308C_{228} + 4812C_{229} - 17472C_{230} - 51790C_{231} - 1344C_{232} \\
& + 4752C_{233} + 15336C_{234} - 325702C_{235} - 5652C_{236} - 18264C_{237} - 39024C_{238} - 266390C_{239} - 35304C_{240} \\
& + 233132C_{241} - 12042C_{242} - 5244C_{243} - 45756C_{244} + 3612C_{245} - 244776C_{246} + 18230C_{247} + 8712C_{248} \\
& + 1932C_{249} - 62988C_{250} - 8436C_{251} - 5952C_{252} - 150596C_{253} - 3864C_{254} - 2136C_{255} - 125024C_{256} \\
& + 1992C_{257} + 1728C_{258} - 38492C_{259} - 194014C_{260} - 2460C_{261} + 28500C_{262} - 4728C_{263} - 37756C_{264} \\
& + 4536C_{265} - 155092C_{266} + 1836C_{267} - 325192C_{268} - 116450C_{269} + 2244C_{270} - 223856C_{271} + 876C_{272} \\
& + 1260C_{273} - 3264C_{274} - 350994C_{275} - 8196C_{276} + 74408C_{277} - 187996C_{278} + 2772C_{279} - 5054C_{280}
\end{aligned}$$

$$\begin{aligned}
& + 1260C_{281} - 97248C_{282} - 3732C_{283} - 183678C_{284} - 7920C_{285} - 271274C_{286} + 8160C_{287} + 42640C_{288} \\
& + 19236C_{289} - 4118C_{290} + 16740C_{291} - 180C_{292} - 16500C_{293} - 299976C_{294} - 16776C_{295} + 431810C_{296} \\
& - 15576C_{297} - 13080C_{298} - 256796C_{299} + 1248C_{300} - 12984C_{301} - 209848C_{302} - 27468C_{303} - 31896C_{304} \\
& - 352402C_{305} - 34152C_{306} + 3024C_{307} - 492C_{308} - 20448C_{309} - 366528C_{310} - 22128C_{311} - 183650C_{312} \\
& + 13392C_{313} + 11052C_{314} + 15660C_{315} + 11340C_{316} - 11700C_{317} - 9540C_{318} - 12744C_{319} - 14664C_{320} \\
& - 16692C_{321} - 288C_{322} - 144C_{323} - 1572C_{324} - 4860C_{325} - 158448C_{326} - 2976C_{327} - 10128C_{328} \\
& - 60114C_{329} - 1152C_{330} - 9816C_{331} - 479106C_{332} - 20484C_{333} - 3156C_{334} - 11148C_{335} - 20892C_{336} \\
& - 258324C_{337} - 34692C_{338} - 144C_{339} - 2940C_{340} + 4956C_{341} - 219758C_{342} - 5244C_{343} - 27420C_{344} \\
& - 58968C_{345} + 3144C_{346} + 3900C_{347} - 372174C_{348} - 780C_{349} + 3528C_{350} - 8352C_{351} - 28008C_{352} \\
& - 209924C_{353} - 30576C_{354} - 19200C_{355} - 18516C_{356} - 208418C_{357} + 2784C_{358} + 312C_{359} - 18420C_{360} \\
& - 207992C_{361} + 2892C_{362} - 31080C_{363} - 37392C_{364} - 145046C_{365} - 49656C_{366} + 3624C_{367} - 5040C_{368} \\
& - 29568C_{369} - 88052C_{370} - 37632C_{371} - 78438C_{372} - 187038C_{373} + 160810C_{374} + 6990C_{375} - 37872C_{376} \\
& - 43702C_{377} + 182330C_{378} + 56778C_{379} - 11448C_{380} - 250466C_{381} - 22364C_{382} - 121402C_{383} + 43126C_{384} \\
& - 4968C_{385} - 2758C_{386} + 265038C_{387} + 231260C_{388} + 288100C_{389} + 3240C_{390} - 18324C_{391} + 194794C_{392} \\
& + 283548C_{393} + 515637C_{394} - 2316C_{395} + 323536C_{396} + 481808C_{397} - 51474C_{398} - 27764C_{399} + 720C_{400} \\
& - 73830C_{401} + 184948C_{402} + 219794C_{403} - 6576C_{404} - 125438C_{405} + 547099C_{406} + 373157C_{407} + 71732C_{408} \\
& + 165186C_{409} - 28358C_{410} + 294458C_{411} + 10008C_{412} + 5328C_{413} - 25298C_{414} + 89932C_{415} - 121304C_{416} \\
& + 322458C_{417} - 35738C_{418} + 202833C_{419} - 31276C_{420} + 726792C_{421} + 9972C_{422} - 2760C_{423} - 94436C_{424} \\
& + 537527C_{425} - 94412C_{426} + 767108C_{427} + 8400C_{428} - 6624C_{429} + 274923C_{430} + 499389C_{431} - 47262C_{432} \\
& + 184238C_{433} + 5520C_{434} + 204C_{435} + 8414C_{436} + 77216C_{437} - 100388C_{438} + 229320C_{439} - 5532C_{440} \\
& + 392222C_{441} - 1212C_{442} + 10068C_{443} - 6012C_{444} - 169735C_{445} - 79048C_{446} + 346939C_{447} - 85488C_{448} \\
& + 584211C_{449} + 1464C_{450} - 4620C_{451} - 24846C_{452} + 237574C_{453} - 46686C_{454} - 197994C_{455} + 121016C_{456} \\
& + 65142C_{457} - 100316C_{458} - 98460C_{459} + 40844C_{460} - 456C_{461} - 5976C_{462} - 85866C_{463} - 14984C_{464} \\
& - 62636C_{465} - 120454C_{466} - 16498C_{467} + 5076C_{468} + 2868C_{469} - 25334C_{470} + 27912C_{471} + 145006C_{472} \\
& + 135304C_{473} - 546915C_{474} - 2172C_{475} - 45760C_{476} + 1080C_{477} - 4764C_{478} - 67712C_{479} + 72894C_{480} \\
& - 48172C_{481} + 106442C_{482} - 26852C_{483} + 6792C_{484} - 564C_{485} - 54540C_{486} + 65725C_{487} - 78148C_{488} \\
& - 45105C_{489} - 56430C_{490} - 196792C_{491} + 6420C_{492} + 4344C_{493} + 16046C_{494} + 3506C_{495} + 518556C_{496} \\
& + 14436C_{497} + 8448C_{498} - 202292C_{499} - 120572C_{500} - 106874C_{501} - 167120C_{502} - 457738C_{503} - 144C_{504} \\
& - 43668C_{505} + 180350C_{506} + 648C_{507} - 4176C_{508} + 8546C_{509} + 49202C_{510} - 54142C_{511} + 290692C_{512} \\
& + 481507C_{513} + 7296C_{514} + 528C_{515} + 556337C_{516} + 675051C_{517} + 11832C_{518} + 5772C_{519} + 236561C_{520} \\
& + 554071C_{521} - 28810C_{522} - 88202C_{523} + 6432C_{524} + 3732C_{525} + 19202C_{526} + 53332C_{527} + 81882C_{528} \\
& + 118434C_{529} + 18048C_{530} + 7872C_{531} - 71070C_{532} - 152514C_{533} - 837964C_{534} + 14904C_{535} + 12516C_{536} \\
& - 48792C_{537} + 139934C_{538} + 333706C_{539} + 195122C_{540} - 5550C_{541} + 2520C_{542} - 9334C_{543} + 12558C_{544} \\
& + 293278C_{545} - 9900C_{546} - 34994C_{547} + 192956C_{548} - 25094C_{549} + 589686C_{550} + 2120C_{551} - 15360C_{552} \\
& - 145954C_{553} - 54654C_{554} - 49338C_{555} - 46614C_{556} - 16744C_{557} + 705212C_{558} - 240C_{559} - 136382C_{560} \\
& + 66116C_{561} - 9728C_{562} - 4672C_{563} - 11484C_{564} - 23148C_{565} + 14558C_{566} + 10302C_{567} + 239250C_{568} \\
& - 15864C_{569} + 417133C_{570} - 22538C_{571} + 77869C_{572} - 5844C_{573} - 171993C_{574} + 201203C_{575} + 8368C_{576} \\
& + 60594C_{577} + 99130C_{578} - 48346C_{579} - 52546C_{580} - 273972C_{581} - 230570C_{582} - 905942C_{583} - 214963C_{584} \\
& + 217439C_{585} + 145604C_{586} - 12708C_{587} + 306690C_{588} + 10198C_{589} + 137228C_{590} + 346058C_{591} + 186122C_{592} \\
& + 158174C_{593} + 31594C_{594} + 157286C_{595} + 270234C_{596} + 295582C_{597} - 78596C_{598} + 123409C_{599} - 671038C_{600}
\end{aligned}$$

$$\begin{aligned}
& + 29136C_{601} + 28520C_{602} + 33346C_{603} - 17632C_{604} + 225875C_{605} + 48128C_{606} - 79590C_{607} + 65096C_{608} \\
& - 94964C_{609} + 149694C_{610} + 486422C_{611} + 506574C_{612} - 85370C_{613} + 480920C_{614} + 3720C_{615} + 798026C_{616} \\
& + 201043C_{617} + 313962C_{618} + 292500C_{619} - 11268C_{620} + 107479C_{621} + 590932C_{622} - 171482C_{623} + 967553C_{624} \\
& - 61746C_{625} + 783790C_{626} + 65073C_{627} + 111508C_{628} + 1202622C_{629} + 30466C_{630} - 163486C_{631} + 1039771C_{632} \\
& + 30178C_{633} + 811337C_{634} + 138379C_{635} - 37184C_{636} - 35378C_{637} + 55370C_{638} - 168724C_{639} - 429016C_{640} \\
& - 79884C_{641} - 155454C_{642} + 7584C_{643} - 231631C_{644} - 485137C_{645} - 39500C_{646} - 658713C_{647}
\end{aligned}$$

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